Zero Energy States for $SU(N)$: A Simple Exercise in Group Theory ?

M. Bordemann
Fakultät für Physik, Uni Freiburg
e-mail: mbor@majestix.physik.uni-freiburg.de

J. Hoppe
Max-Planck-Institut für Gravitationsphysik
e-mail: hoppe@aei-potsdam.mpg.de

R. Suter
Mathematik Departement ETH Zürich
e-mail: suter@math.ethz.ch

Abstract

We show that the requirement of $S_3 \times Spin(d)$ invariance for an "asymptotically free" $SU(3)$-Cartan subalgebra wave-function does *not* give a unique candidate for a $SU(3)$-invariant zero energy state of the $d = 9$ supersymmetric matrix model– nor does it rule out the existence of such a state in the case $d = 3$.

Due to coherent evidence[1] that, asymptotically, a zero energy state of a $SU(N)$-invariant supersymmetric matrix model in $d = 3,\ 5$ or 9 dimensions factorizes into a part involving only the Cartan subalgebra degrees of freedom (with effectively free dynamics) and a part forming supersymmetric harmonic oscillators, the following "wishful conjecture" appeared:

CW: For each $N \geq 2$, and $d = 9$, the free Laplacian (times the unit operator in a $2^{(N-1)(d-1)}$-dimensional Fock space) admits exactly one $Spin(d) \times S_N$ invariant wavefunction (where S_N denotes the permutation group of N letters) which is square integrable at ∞ and harmonic everywhere except at the origin, –whereas for $d = 5$ and 3 no such function exists.

If true, **CW** would provide further evidence for the global existence of a unique (normalizable) zero energy state for $d = 9$, and (assuming effectively free dynamics for the Cartan subalgebra part) prove the nonexistence of a global zero energy state for $d = 3$ and 5.

For $N = 2$, the $d = 9$ part of **CW** is implied by [4], [5], while the nonexistence for $d = 3$ and 5 was proven in [6], [3].

In this note we shall prove **CW** to be *wrong* (for $N = 3$), by explicit construction of five linearly independent $S_3 \times Spin(9)$-invariant, and one $S_3 \times Spin(3)$-invariant, wave-functions.

For $N = 3$, the ingredients are the tensor product of two copies of a 2^{d-1}-dimensional Fock space $\mathcal{H}$, each decomposing as

$$\mathcal{H}_4 = \{1\} \oplus \{1\} \oplus \{2\} \tag{1}$$

$$\mathcal{H}_{16} = \{1\} \oplus \{1\} \oplus \{1\} \oplus \{5\} \oplus \{4\} \oplus \{4\} \tag{2}$$

$$\mathcal{H}_{256} = \{44\} \oplus \{84\} \oplus \{128\} \tag{3}$$

under $Spin(d)$, respectively (where we write $\{n\}$ for an irreducible representation space of dimension n), and the space of all harmonic polynomials in $2d$ variables $z = (x, y)$ where $z_a =: x_a$ for $a \leq d$ and $z_a =: y_{a-d}$ for $a > d$. An arbitrary harmonic polynomial of degree l takes the form

$$h(z) = \sum_{1 \leq a_1 \cdots a_l \leq 2d} c_{a_1 \cdots a_l} z_{a_1} \cdots z_{a_l} \tag{4}$$

[1]See e.g. [1] and references therein (note also [2]); a complete derivation for $SU(2)$ can be found in [3].

where the tensor c is totally symmetric and traceless between any two indices. As (setting $r := \sqrt{\sum_a (z_a)^2}$)

$$((\frac{\partial}{\partial r})^2 + \frac{2d-1}{r}\frac{\partial}{\partial r} - \frac{l(l+2d-2)}{r^2})r^{-l-(2d-2)} = 0, \tag{5}$$

each homogeneous harmonic polynomial of degree l may be multiplied by $r^{-2l-(2d-2)}$ (to ensure a square integrable fall-off at ∞), without losing harmonicity away from the origin.

The Weyl group for $SU(N)$ is known to be the symmetric group S_N,– which for $N = 3$ has only three pairwise inequivalent irreducible representations, namely the trivial representation in a one-dimensional module (denoted by $\{1\}$), the alternating representation by the sign of the permutations in a one-dimensional module (denoted by ϵ), and the standard representation in a two dimensional module (denoted by ρ) which in some orthonormal basis takes the following form:

$$\begin{array}{lll} 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & C = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, & C^2 = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \\ P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & P' = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, & P'' = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \end{array} \tag{6}$$

where C and C^2 denote the two nontrivial cyclic permutations and P, P', P'' denote the three transpositions. Taking traces, the characters of the above irreducible representations are easily computed to take the following values on the three conjugacy classes $\{1\}$, $\{C, C^2\}$, and $\{P, P', P''\}$:

$$\begin{array}{lcccc} \text{for } 1 & : & 1, & 1, & 1 \\ \text{for } \epsilon & : & 1, & 1, & -1 \\ \text{for } \rho & : & 2, & -1, & 0. \end{array} \tag{7}$$

In the sequel we shall identify the characters with the equivalence class of irreducible representations they define.

The generators of (the Lie algebra of) $Spin(d)$ can be represented in fermionic Fock space $\wedge\mathbb{C}^{(d-1)(N-1)}$ as operators in the following way where $s_d := 4, 8, 16$ for $d = 3, 5, 9$):

$$\begin{array}{rcl} M_{d,d-1} & = & \frac{i}{2}\big(\mu_{\alpha n}\partial_{\mu_{\alpha n}} - \frac{N-1}{4}s_d\big) \\ M_{d,j} & = & \frac{1}{4}\Gamma^j{}_{\alpha\beta}\big(\mu_{\alpha n}\mu_{\beta n} - \partial_{\mu_{\alpha n}}\partial_{\mu_{\beta n}}\big) \\ M_{d-1,j} & = & \frac{-i}{4}\Gamma^j{}_{\alpha\beta}\big(\mu_{\alpha n}\mu_{\beta n} + \partial_{\mu_{\alpha n}}\partial_{\mu_{\beta n}}\big) \\ M_{jk} & = & \frac{1}{2}\Gamma^{jk}{}_{\alpha\beta}\mu_{\alpha n}\partial_{\mu_{\beta n}} \end{array} \tag{8}$$

Here the $\mu_{\alpha m}$ and $\partial_{\mu_{\beta n}}$ $(1 \leq \alpha, \beta \leq d-1,\ 1 \leq m, n \leq N-1)$ are fermionic creation (left exterior multiplication) and annihilation (interior product) operators, $1 \leq j, k \leq d-2$, the $(d-2)$ Γ^j are purely imaginary, antisymmetric matrices satisfying the anticommutation relations

$$\{\Gamma^j, \Gamma^k\} = 2\delta^{jk} 1_{(d-1)\times(d-1)}$$

and the $\Gamma^{jk}_{\alpha\beta}$ are the commutators $\frac{1}{2}[\Gamma^j, \Gamma^k]_{\alpha\beta}$. For $N = 3$ we denote $\mu_{\alpha 1}$ by λ_α and $\mu_{\alpha 2}$ by μ_α.

For $d = 3$, $\mathcal{H}_4 \otimes \mathcal{H}_4$ gives five $Spin(3)$-singlets, four 2-dimensional $Spin(3)$-representations, and one 3-dimensional one spanned by

$$1, \lambda_1\lambda_2 + \mu_1\mu_2, \lambda_1\lambda_2\mu_1\mu_2. \tag{9}$$

While the one-dimensional spaces spanned by $\lambda_1\mu_1$, $\lambda_2\mu_2$, and $\lambda_1\mu_2 + \lambda_2\mu_1$, respectively, are isomorphic to ϵ under S_3, the remaining two $Spin(3)$-singlets span a S_3-module isomorphic to ρ with basis

$$|1\rangle := \frac{1}{\sqrt{2}}(\mu_1\lambda_2 - \mu_2\lambda_1), \quad |2\rangle := \frac{1}{\sqrt{2}}(\lambda_1\lambda_2 - \mu_1\mu_2). \tag{10}$$

On the other hand one can easily check that the two $SO(3)$-scalars $\vec{x}^2 - \vec{y}^2$ and $2\vec{x}\cdot\vec{y}$ transform the same way under the two transpositions P and P' (hence, under S_3) as $|1\rangle$ and $|2\rangle$. Therefore

$$\psi_{d=3}(\vec{x}, \vec{y}) := \frac{1}{r^8}\left((\vec{x}^2 - \vec{y}^2)(\lambda_1\lambda_2 - \mu_1\mu_2) + 2\vec{x}\cdot\vec{y}(\mu_1\lambda_2 - \mu_2\lambda_1)\right) \tag{11}$$

is invariant under $Spin(3) \times S_3$, as well as asymptotically normalizable

$$\int_\Lambda^\infty r^5 dr r^{-12} \quad < \quad \infty$$

and harmonic

$$\Delta_{\mathbb{R}^6}\psi_{d=3} = 0, \tag{12}$$

thus disproving the $d = 3$ part of **CW**.

For $d = 9$, things are somewhat more complicated: while the decompositions into irreducible $Spin(9)$-modules in the tensor product of two copies of the fermionic Fock space can be computed using the formulas (where we agree upon writing $m\{n\}$ for the direct sum of m copies of the irreducible module $\{n\}$)[2].

$$\begin{aligned}
\{44\} \otimes \{44\} &= \{1\} \oplus \{36\} \oplus \{44\} \oplus \{450\} \oplus \{495\} \oplus \{910\} && (13)\\
\{44\} \otimes \{84\} &= \{84\} \oplus \{231\} \oplus \{924\} \oplus \{2457\} && (14)\\
\{84\} \otimes \{84\} &= \{1\} \oplus \{36\} \oplus \{44\} \oplus \{84\} \oplus \{126\} \oplus \{495\} \oplus \{594\}\\
&\quad \oplus \{924\} \oplus \{1980\} \oplus \{2772\} && (15)\\
\{44\} \otimes \{128\} &= \{16\} \oplus \{128\} \oplus \{432\} \oplus \{576\} \oplus \{1920\} \oplus \{2560\}\\
& && (16)\\
\{84\} \otimes \{128\} &= \{16\} \oplus 2\{128\} \oplus 2\{432\} \oplus \{567\} \oplus \{672\} \oplus \{768\}\\
&\quad \oplus \{2560\} \oplus \{5040\} && (17)\\
\{128\} \otimes \{128\} &= \{1\} \oplus \{9\} \oplus 2\{36\} \oplus \{44\} \oplus 2\{84\} \oplus 2\{126\}\\
&\quad \oplus \{156\} \oplus 2\{231\} \oplus \{495\} \oplus 2\{594\} \oplus \{910\}\\
&\quad \oplus 2\{924\} \oplus \{1650\} \oplus \{2457\} \oplus \{2772\} \oplus \{3900\} && (18)
\end{aligned}$$

found in the literature (see e.g. [7, p.103, Table 40]) the individual transformation properties under S_3 are more difficult to obtain: according to the general theory of compact Lie groups this would amount to a finer decomposition of the above space into tensor products of irreducible S_3-modules with irreducible $Spin(9)$-modules; note however that there is only one (!) 9-dimensional representation on the right hand sides of the previous six equations (which, therefore, must be equivalent either to $\{1\} \otimes \{9\}$ or to $\epsilon \otimes \{9\}$).

Calculating the decomposition into irreducible $S_3 \times Spin(9)$-submodules of the space of all cubic harmonic polynomials one finds

$$\begin{aligned}
Sym^3_{\mathrm{harm}} &= \{1\} \otimes \{9\} \oplus \epsilon \otimes \{9\} \oplus \rho \otimes \{9\} \oplus \{1\} \otimes \{156\}\\
&\quad \oplus \epsilon \otimes \{156\} \oplus \rho \otimes \{156\} \oplus \rho \otimes \{231\}. && (19)
\end{aligned}$$

To obtain this, we first used the decomposition into irreducible $Spin(9)$-

[2] In this note we shall not encounter inequivalent irreducible $Spin(d)$-modules of the same dimension.

submodules

$$
\begin{aligned}
Sym^3(\mathbb{R}^9 \oplus \mathbb{R}^9) &= 2\{Sym^3(\mathbb{R}^9)\} \oplus 2\{Sym^2(\mathbb{R}^9) \otimes \mathbb{R}^9\} \\
&= 2\{156 \oplus 9\} \oplus 2\{\{1 \oplus 44\} \otimes 9\} \\
&= 6\{9\} \oplus 4\{156\} \oplus 2\{231\} \qquad (20) \\
& \qquad (21)
\end{aligned}
$$

as well as the decomposition into irreducible S_3-submodules (using $\rho \otimes \rho = \{1\} \oplus \epsilon \oplus \rho$, see the character table (7)):

$$
Sym^3(\mathbb{R}^9 \oplus \mathbb{R}^9) = 165\{1\} \oplus 165\epsilon \oplus 405\rho \tag{22}
$$

where these have to be distributed among (20) (which turns out to be doable by just counting dimensions); finally, one of the emerging $\rho \otimes \{9\}$ modules had to be dropped, as it corresponds to the submodule spanned by the $r^2 x_a$ and $r^2 y_a$ $(1 \leq a \leq 9)$ which does not contain any nonzero harmonic functions. In any case, as (19) contains both $\{1\} \otimes \{9\}$ and $\epsilon \otimes \{9\}$, the single $Spin(9)$-submodule of dimension 9 in the fermionic sector (18) –no matter whether it is of type $\{1\} \otimes \{9\}$ or $\epsilon \otimes \{9\}$–can be matched in the final tensor products of bosons and fermions to form a $S_3 \times Spin(9)$ invariant wave function of the form

$$
\Psi(z) = r^{-22} \sum_{s=1}^{9} \psi_s(z) |9; s\rangle \tag{23}
$$

where the $|9; s\rangle$ form an orthonormal basis of the $\{1\} \otimes \{9\}$-module in (18) and the ψ_s form an orthonormal basis of the module of the same type in the space of harmonic cubic polynomials.

One could be tempted to deduce that this must be the Cartan subalgebra factor in the asymptotic form of *the* $d = 9$, $N = 3$ zero energy wave-function. However, at least (!) four other $S_3 \times Spin(9)$ candidate wave-functions exist (thus disproving the uniqueness of the $d = 9$ part of **CW**); to see this, replace 9 by 156 in the previous argument (which gives the second invariant ground state), or consider the space of all quartic harmonic polynomials, which after some work can be concluded to decompose into

$$
\begin{aligned}
Sym^4_{\mathrm{harm}}(\mathbb{R}^{18}) = \; & \{1\} \otimes \{450\} \oplus 2\rho \otimes \{450\} \oplus \rho \otimes \{910\} \oplus \epsilon \otimes \{910\} \\
& \oplus \{1\} \otimes \{495\} \oplus \{1\} \otimes \{44\} \oplus \epsilon \otimes \{44\} \oplus 2\rho \otimes \{44\} \\
& \oplus \rho \otimes \{36\} \oplus \rho \otimes \{1\} \oplus \{1\} \otimes \{1\}
\end{aligned} \tag{24}
$$

under $S_3 \times Spin(9)$.

Now note that the r.h.s of eqs (13)-(18) contains three $Spin(9)$-submodules of dimension 495. Now the submodule of dimension 128 in (3) consists of all forms of odd degree over $\mathbb{R}^8$ (see [5]) whereas the two submodules of dimension 44 and 84 give all corresponding forms of even degree. It follows that the transposition $P \in S_3$, which maps (λ_a, μ_a) to $(\lambda_a, -\mu_a)$, is equal to the identity on the two irreducible modules of dimension 495 contained in (13) and (15) since they imply even forms in λ and μ separately, whereas it is equal to minus identity on the irreducible module of dimension 495 contained in (18) since it consists of forms which are odd both in λ and in μ. Out of the 6 apriori possibilities to decompose the sum of the three modules of dimension 495 into irreducibles under $S_3 \times Spin(9)$, the two *not* containing $\{1\} \otimes \{495\}$, namely $\epsilon \otimes \{495\} \ \oplus \ \rho \otimes \{495\}$ and $3\epsilon \otimes \{495\}$, are therefore impossible, as their $+1$-eigenspace under P would be too small. Therefore the fermionic sector contains at least one $S_3 \times Spin(9)$-submodule of type $\{1\} \otimes \{495\}$ that can be matched with the irreducible submodule of dimension 495 in (24) in the same manner as in the cubic case (see (23), viz.

$$\Psi(z) = r^{-24} \sum_{s=1}^{495} \psi_s(z) |495; s\rangle \tag{25}$$

where the $|495; s\rangle$ form an orthonormal basis of the $\{1\} \otimes \{495\}$-module in the fermionic sector and the ψ_s form an orthonormal basis of the module of the same type in the space of harmonic quartic polynomials. The ψ_s can all be chosen out of the quartic polynomials taking the following form:

$$\psi_s(x, y) = \sum c^{(s)}{}_{a_1 a_2 a_3 a_4} (x_{a_1} x_{a_2} - \frac{1}{9} x_a x_a \delta_{a_1 a_2})(y_{a_3} y_{a_4} - \frac{1}{9} y_a y_a \delta_{a_3 a_4}) \tag{26}$$

where $c^{(s)}$ is a totally symmetric tensor of rank four in $\mathbb{R}^9$. This will give a third candidate for the Cartan subalgebra factor of an asymptotic zero-energy state of the $d = 9$ $SU(3)$ matrix model.

A fourth candidate is obtained by noting that the decomposition (24) contains the three pairwise nonequivalent irreducible $S_3 \times Spin(9)$-modules $\{1\} \otimes \{44\}$, $\epsilon \otimes \{44\}$, and $\rho \otimes \{44\}$. According to eqs (13), (15), and (18) the fermionic sector contains one of these modules which again match to a $S_3 \times Spin(9)$-invariant state of the form

$$\Psi(z) = r^{-24} \sum_{s=1}^{44} \psi_s(z) |44; s\rangle \tag{27}$$

with the obvious notations.

Last, but not the least, consider the space of all quadratic harmonic polynomials which decomposes as

$$Sym^2_{\rm harm} = \rho \otimes \{1\} \oplus \rho \otimes \{44\} \oplus \{1\} \otimes \{44\} \oplus \epsilon \otimes \{36\} \tag{28}$$

under $S_3 \times Spin(9)$ (corresponding to the space spanned by $\vec{x}^2 - \vec{y}^2$ and $2\vec{x} \cdot \vec{y}$, the space of polynomials $\sum c_{ab}(x_a x_b - y_a y_b)$ and $\sum c_{ab}(x_a y_b)$ with c symmetric traceless, the space of polynomials $\sum c_{ab}(x_a x_b + y_a y_b)$ with c symmetric traceless, and the space of polynomials $\sum d_{ab}(x_a y_b)$ with d antisymmetric, respectively). Again, one of the above modules containing $\{44\}$ as a factor can be matched with an equivalent fermionic one, as at least one of the two fermionic modules containing $\{44\}$ as a factor is *not* isomorphic to $\epsilon \otimes \{44\}$: otherwise all the three $Spin(9)$-modules of type $\{44\}$ in (13), (15), and (18) would have to change sign under the transposition P which is not the case for the two modules of type $\{44\}$ in (13) and (15), compare the discussion of $\{495\}$.

Finally, note that while $H\psi = 0$ implies $Q\psi = 0$ if Q is hermitean and $H = Q^2$, care is needed when looking at the corresponding differential equations only asymptotically, or when deriving 'effective' operators (as done in [1]). As stressed by A.Smilga [8], it may well be that the additional condition $Q_{\rm eff}\psi = 0$ will single out a unique $d = 9$ wave-function and/or exclude any $d = 3, 5$ $S_N \times Spin(d)$-wave-function. It should be easy to test this using our $S_3 \times Spin(d)$-invariant wave-functions.

Acknowledgements: J.H. would like to thank J.Plefka and A.Smilga for valuable discussions, while M.B. thanks the Albert-Einstein-Institut for friendly hospitality.

References

[1] Kac, V.G., Smilga, A.V.: `hep-th/9908096`.

[2] Konechny, A.: *On asymptotic Hamiltonian for $SU(N)$ matrix theory*, JHEP 9810 (1998); `hep-th/9805046`.

[3] Fröhlich, J., Graf, G.M., Hasler, D., Hoppe, J., Yau, S.-T.: *Asymptotic Form of Zero Energy Wave Functions in Supersymmetric Matrix Models*, `hep-th/9904182`.

[4] Halpern, M., Schwartz, C.: *Asymptotic search for ground states of $SU(2)$ matrix theory*, Int. J. Mod. Phys. A**13**, 4367-4408 (1998); `hep-th/9712133`.

[5] Graf, G.M., Hoppe,, J.: *Asymptotic ground state for 10 dimensional reduced supersymmetric $SU(2)$ Yang Mills Theory*, hep-th/9805080.

[6] Hoppe, J., Yau, S.-T.: *Absence of zero energy states in the simplest $d = 3$ ($d = 5$?) matrix models*, `hep-th/9806152`.

[7] Slansky, R.: *Group Theory for Unified Model Building*, Phys. Reports **79** (1981), 1-128.

[8] Smilga, A.: private communication.